THE UNIVERSITY OF WESTERN ONTARIO
FACULTY OF ENGINEERING SCIENCE
DEPARTMENT OF ELECTRICAL ENGINEERING

E.S. 760b COMPUTATIONAL ELECTROMAGNETICS

Final Examination - May 3 - 13, 1996.

Time allotted: one week take home exam, papers are to be returned by 4:00 on May 13.

General Instructions:

1) This is a **take home** exam.

2) There are **7** questions in total.

3) Answer **all** questions as completely as possible.

4) All questions have equal weighting.

5) Print clearly.

6) Clearly indicate all the assumptions you've made in your solutions.

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1) The linear system of equations

$$\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix} \mathbf{x} = \mathbf{b}$$

where a is real, can under certain conditions be solved by the iterative method

$$\begin{bmatrix} 1 & 0 \\ -\omega a & 1 \end{bmatrix} \mathbf{x}^{(k+1)} = \begin{bmatrix} 1-\omega & \omega a \\ 0 & 1-\omega \end{bmatrix} \mathrm{x}^{(k)} + \omega \mathbf{b}.$$

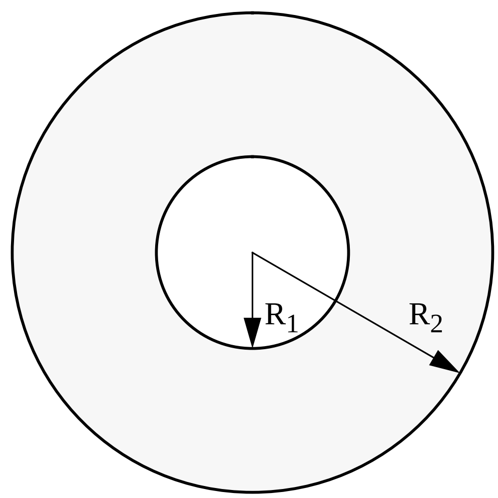
(a) For which values of a is the method convergent for $\omega = 1$.

(b) For $a = 0.5$, find the value of $\omega \in \{0.8, 0.9, 1.0, 1.1, 1.2, 1.3\}$ which minimizes the spectral radius of the matrix

$$\begin{bmatrix} 1 & 0 \\ -\omega a & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1-\omega & \omega a \\ 0 & 1-\omega \end{bmatrix}$$

and explain why we would want to do this.

2) Consider two coaxial cylinders shown in the figure. The space between cylinders is filled with non-uniformly charged dielectric material $\rho(r)=\rho_0 r/R_1$ (relative permeability $\varepsilon_r=2.0$, $\varepsilon_0=8.85\mathrm{x}10^{-12}$[F/m], $\rho_0=10^{-6}$ [C/m^2]). R_1= 1 [cm], R_2 = 10 [cm].



a) Solve the problem analytically by integrating Poisson's equation.

b) Discretize the domain into three equal finite elements.

c) Assuming linear interpolation, derive the matrix equation for each element.

d) Assemble the set of algebraic equations for the nodal values of solution using the finite element method.

e) Introduce boundary conditions.

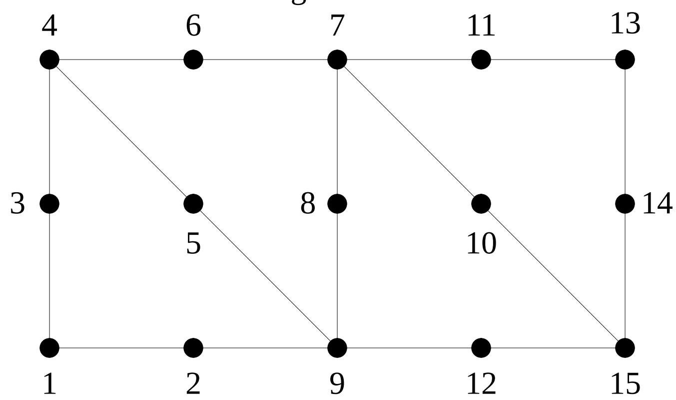
f) Solve the set of algebraic equations.

g) Compare analytical and numerical solutions at nodes

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)$$

3) Show that the one-dimensional time domain Maxwell's equations cannot be uncouple by diagonalization techniques for the case of a conducting medium ($\sigma \neq 0$).

4) For the following finite element mesh assemble the matrix [S], assuming the second order approximation of a solution. Each triangle has one 90^o and two 45^o angles.



5) Show that Yee's algorithm for solving Maxwell's equations can be derived by starting from the integral form of Maxwell's equations instead of from the differential form. Derive all six explicit update equations.

6) Derive the 2nd order Mur absorbing boundary conditions for the 3-D case. Choose one boundary plane to derive the explicit update equations (*e.g.* $x = 0$ plane). Is it possible to derive different difference equations starting from the same partial differential equation which is derived by Mur for the boundary $x = 0$?

7) Consider the one-dimensional Poisson's equation, boundary value problem given as

ODE: $\dfrac{d^2}{dx^2}u(x) = xe^x \qquad 0 \le x \le \pi$ B.C.'s: $u(0) = 0,\ u(\pi) = 0.$

(a) Show that the exact solution is given by

$$u(x) = xe^x - 2e^x + \left(\frac{2e^\pi}{\pi} - e^\pi - \frac{2}{\pi}\right)x + 2 \qquad 0 \le x \le \pi.$$

(b) Use Galerkin's method (*i.e.* specific instance of the Method of Moments) with basis functions given as

basis functions: $u_n = \sin(nx), \qquad n = 1, 2, 3, \ldots, N$

to create an approximate expansion of the form

$$u(x) = \sum_{n=1}^{N} \alpha_n u_n = \underline{u}^T\underline{\alpha} = \underline{\alpha}^T\underline{u}$$

and formulate the matrix equation which approximates the solution to the above problem. Derive the components of the matrix equation for any size expansion (*i.e.* any N). Find the approximate solution for the case N = 3. Use the inner product defined as

inner product: $(f, g) = \displaystyle\int_0^\pi fg\,dx.$

(c) What do you notice about the structure of the matrix and why is it of this special form?